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Tri-bimaximal lepton mixing from symmetry only

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Abstract: We construct a model for tri-bimaximal lepton mixing which employs only family symmetries and their soft breaking; neither vacuum alignment nor supersymmetry, extra dimensions, or non-renormalizable terms are used in our model. It is an extension of the Standard Model making use of the seesaw mechanism with five right-handed neutrino singlets. The scalar sector comprises four Higgs doublets and one complex gauge singlet. The horizontal symmetry of our model is based on the permutation group S_3 of the lepton families together with the three family lepton numbers — united this constitutes a symmetry group $\Delta(6\infty^2)$. The model makes no predictions for the neutrino masses.

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1 Introduction

Lepton mixing and non-zero neutrino masses are now established facts — for reviews and for the latest fits see [1]. The mixing angles in the lepton mixing matrix U have values quite different from those of quark mixing. The phenomenological hypothesis that

$$U = U_{\mathrm{HPS}} \equiv \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \tag{1.1}$$

has been put forward by Harrison, Perkins and Scott (HPS) in 2002 [2]. At present, all the experimental data are still compatible with this simple "tri-bimaximal" mixing *Ansatz*.

The hypothesis (1.1) has stimulated model building and the search for family symmetries which might lead to $U = U_{\mathrm{HPS}}$ in a natural way. While it is not difficult to simultaneously obtain $U_{e3} = 0$ and maximal atmospheric-neutrino mixing [3], generating a solar mixing angle $\theta_{12} = \arcsin\left(1/\sqrt{3}\right)$ is highly non-trivial and in general necessitates complicated models. In those models one often finds several scalar multiplets of the horizontal-symmetry group with vacuum expectation values (VEVs) aligned in a special way. To explain this peculiar alignment of VEVs one may have recourse to special scalar potentials, stabilized with the help of supersymmetry — see for instance [4–6] — or to extra-dimensional models [7].

In two previous papers [8, 9] we have enforced trimaximal mixing — which is a weaker hypothesis than tri-bimaximal mixing — through a model. We now show that, with very little extra effort, one can also achieve tri-bimaximal mixing along the same lines. In the model that we shall present here neither VEV alignment nor supersymmetry, non-renormalizable terms, or extra dimensions are required for obtaining $U = U_{\mathrm{HPS}}$. Besides enlarging the scalar sector of the Standard Model (SM) by several Higgs doublets and one gauge singlet, our model uses the seesaw mechanism [10] with *more than three* right-handed neutrino singlets, but in such a way that the additional right-handed neutrinos do not have Yukawa couplings to the Higgs doublets; then these additional right-handed neutrinos — in the present case there are two of them — can be exploited for imposing the desired mixing properties.[1] In our model lepton mixing originates solely in the Majorana mass matrix M_R of the right-handed neutrino singlets, and the number of independent Yukawa coupling constants of the Higgs doublets is an absolute minimum — only two.

This paper is organized as follows. The model is presented in section 2. Variations on the symmetries of the model, and their connection to the renormalization-group evolution (RGE) of the light-neutrino mass matrix $\mathcal{M}_\nu$, are investigated in section 3. The conclusions are presented in section 4. An appendix contains details of the computation of the 3×3 matrix $\mathcal{M}_\nu$ out of the 5×5 matrix M_R.

2 The model

2.1 Fields and symmetries

Our model is based on the SM gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$. The lepton sector[2] consists of three left-handed SU(2) doublets $D_{\alpha L} = (\nu_{\alpha L}, \alpha_L)^T$ ($\alpha = e, \mu, \tau$), three right-handed charged-lepton SU(2) singlets α_R, and *five* right-handed $\mathrm{SU}(2) \times \mathrm{U}(1)$ singlet neutrinos $\nu_{\alpha R}$, $\nu_{\ell R}$ ($\ell = 1, 2$). The scalar sector consists of one complex gauge singlet χ with zero electric charge and four Higgs doublets $\phi_\alpha = \left(\phi_\alpha^+, \phi_\alpha^0\right)^T$, $\phi_0 = \left(\phi_0^+, \phi_0^0\right)^T$.

The family symmetries of the model are the following:

- Three U(1) symmetries associated with the family lepton numbers L_α,

$$\mathrm{U}(1)_{L_\alpha}: \quad D_{\alpha L} \to e^{i\psi_\alpha} D_{\alpha L},\ \alpha_R \to e^{i\psi_\alpha} \alpha_R,\ \nu_{\alpha R} \to e^{i\psi_\alpha} \nu_{\alpha R}, \quad \psi_\alpha \in [0, 2\pi[\,. \tag{2.1}$$

  The $\mathrm{U}(1)_{L_\alpha}$ are supposed to be *softly* broken at high energy, i.e. at the seesaw scale [3, 11], by *dimension-three* terms of the types $\nu_{\alpha L}^T C^{-1} \nu_{\beta L}$, $\nu_{\alpha L}^T C^{-1} \nu_{\ell L}$ (C is the Dirac-Pauli charge-conjugation matrix).

- The S_3 permutation symmetry of the e, μ, τ indices. We view this permutation symmetry as being generated by two non-commuting transformations:

[1]This idea had already been previously used by us for tri-bimaximal mixing, but in that case we still needed VEV alignment and made use of supersymmetry [5].

[2]We neglect the quark sector, which is immaterial for our purposes.

– The cyclic transformation

$$C_{e\mu\tau}: \begin{cases} D_{eL} \to D_{\mu L} \to D_{\tau L} \to D_{eL}, \\ e_R \to \mu_R \to \tau_R \to e_R, \\ \nu_{eR} \to \nu_{\mu R} \to \nu_{\tau R} \to \nu_{eR}, \\ \phi_e \to \phi_\mu \to \phi_\tau \to \phi_e, \\ \nu_{1R} \to \omega\nu_{1R},\ \nu_{2R} \to \omega^2\nu_{2R}, \\ \chi \to \omega\chi,\ \chi^* \to \omega^2\chi^*, \end{cases} \tag{2.2}$$

where $\omega \equiv \exp(2i\pi/3)$ is the cubic root of unity with the properties $\omega^2 = \omega^*$ and $1 + \omega + \omega^2 = 0$.

– The μ–τ interchange [3]

$$I_{\mu\tau}: \begin{cases} D_{\mu L} \leftrightarrow D_{\tau L}, \\ \mu_R \leftrightarrow \tau_R, \\ \nu_{\mu R} \leftrightarrow \nu_{\tau R}, \\ \phi_\mu \leftrightarrow \phi_\tau, \\ \nu_{1R} \leftrightarrow \nu_{2R}, \\ \chi \leftrightarrow \chi^*. \end{cases} \tag{2.3}$$

It is clear that the fields with α indices form triplet *reducible* representations of S_3, while

$$\begin{pmatrix} \nu_{1R} \\ \nu_{2R} \end{pmatrix}, \quad \begin{pmatrix} \chi \\ \chi^* \end{pmatrix}$$

transform under S_3 according to the complex version of the doublet irreducible representation, previously used for instance in [12].[3] The cyclic transformation $C_{e\mu\tau}$ is softly broken by *dimension-two* and *dimension-one* terms in the scalar potential, but it is *preserved* by all the *dimension-three* (and, of course, dimension-four) terms in the Lagrangian. The symmetry $I_{\mu\tau}$ is *not* allowed to be softly broken. The VEV $v_\chi \equiv \langle\chi\rangle_0$ breaks $C_{e\mu\tau}$ spontaneously, but it preserves $I_{\mu\tau}$ because it is *real*; this is a consequence of the $I_{\mu\tau}$-invariance of the scalar potential, as will be shown in subsection 2.3. At low energy, both $C_{e\mu\tau}$ and $I_{\mu\tau}$ are spontaneously broken because all three vacuum expectation values (VEVs) $v_\alpha \equiv \langle\phi_\alpha^0\rangle_0$ are different (see below).

- Three $\mathbb{Z}_2$ symmetries [5, 13]

$$\mathbb{Z}_2^{(\alpha)}: \quad \alpha_R \to -\alpha_R,\ \phi_\alpha \to -\phi_\alpha, \tag{2.4}$$

for $\alpha = e, \mu, \tau$. The $\mathbb{Z}_2^{(\alpha)}$ are supposed to be *softly* broken at low energy, i.e. at the electroweak scale, by *dimension-two* terms of the types $\phi_\alpha^\dagger\phi_\beta$ $(\alpha \neq \beta)$, $\phi_\alpha^\dagger\phi_0$. The symmetry $\mathbb{Z}_2^{(\alpha)}$ is spontaneously broken when ϕ_α^0 acquires the non-zero VEV v_α.

[3] If one wishes one may separate χ into its real and imaginary parts, which transform under S_3 according to the real version of the doublet irreducible representation.

2.2 Lagrangian and lepton mixing

The Yukawa Lagrangian has dimension four and therefore respects all the symmetries of the model. It is given by

$$\mathcal{L}_{\text{Yukawa}} = -y_1 \sum_{\alpha=e,\mu,\tau} \bar{D}_{\alpha L} \alpha_R \phi_\alpha \tag{2.5a}$$

$$-y_2 \sum_{\alpha=e,\mu,\tau} \bar{D}_{\alpha L} \nu_{\alpha R} \left(i\tau_2 \phi_0^* \right) \tag{2.5b}$$

$$+\frac{y_3}{2} \left(\chi\, \nu_{1R}^T C^{-1} \nu_{1R} + \chi^*\, \nu_{2R}^T C^{-1} \nu_{2R} \right) + \text{H.c.} \tag{2.5c}$$

The symmetries $\mathbb{Z}_2^{(\alpha)}$ are instrumental in ensuring that only the doublet ϕ_α couples to α_R — line (2.5a) — and that only the doublet ϕ_0 couples to the three $\nu_{\alpha R}$ — line (2.5b). The family-lepton-number symmetries $\text{U}(1)_{L_\alpha}$ are also important to enforce Yukawa couplings diagonal in flavour space [3]. Note that the number of Yukawa coupling constants of the Higgs doublets is an absolute minimum — just y_1 and y_2.

Upon spontaneous symmetry breaking (SSB) the charged leptons acquire masses $m_\alpha = |y_1 v_\alpha|$. Since those three masses are supposed to be all different, the scalar potential must be rich enough that the VEVs v_α turn out to be all different. Also upon SSB the neutrinos acquire, from line (2.5b), Dirac mass terms

$$- \begin{pmatrix} \bar{\nu}_{eR} & \bar{\nu}_{\mu R} & \bar{\nu}_{\tau R} & \bar{\nu}_{1R} & \bar{\nu}_{2R} \end{pmatrix} M_D \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} + \text{H.c.}, \tag{2.6}$$

where

$$M_D = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \equiv y_2^* v_0, \quad v_0 \equiv \left\langle \phi_0^0 \right\rangle_0 . \tag{2.7}$$

In the Lagrangian there are also bare neutrino Majorana mass terms. These terms have dimension three and are, therefore, allowed to break the family lepton numbers, but not the permutation symmetry S_3. They are

$$\mathcal{L}_{\text{Majorana}} = \frac{M_0^*}{2} \sum_{\alpha=e,\mu,\tau} \nu_{\alpha R}^T C^{-1} \nu_{\alpha R} \tag{2.8a}$$

$$+M_1^* \left(\nu_{eR}^T C^{-1} \nu_{\mu R} + \nu_{\mu R}^T C^{-1} \nu_{\tau R} + \nu_{\tau R}^T C^{-1} \nu_{eR} \right) \tag{2.8b}$$

$$+M_2^* \left[\nu_{1R}^T C^{-1} \left(\nu_{eR} + \omega \nu_{\mu R} + \omega^2 \nu_{\tau R} \right) + \nu_{2R}^T C^{-1} \left(\nu_{eR} + \omega^2 \nu_{\mu R} + \omega \nu_{\tau R} \right) \right] \tag{2.8c}$$

$$+M_4^* \nu_{1R}^T C^{-1} \nu_{2R} + \text{H.c.} \tag{2.8d}$$

Together with line (2.5c) upon SSB, $\mathcal{L}_{\text{Majorana}}$ generates the neutrino Majorana mass terms

$$-\frac{1}{2}\left(\bar{\nu}_{eR}\ \bar{\nu}_{\mu R}\ \bar{\nu}_{\tau R}\ \bar{\nu}_{1R}\ \bar{\nu}_{2R}\right) M_R C \begin{pmatrix} \bar{\nu}_{eR}^T \\ \bar{\nu}_{\mu R}^T \\ \bar{\nu}_{\tau R}^T \\ \bar{\nu}_{1R}^T \\ \bar{\nu}_{2R}^T \end{pmatrix} + \text{H.c.}, \tag{2.9}$$

where the symmetric matrix M_R is

$$M_R = \begin{pmatrix} M_0 & M_1 & M_1 & M_2 & M_2 \\ M_1 & M_0 & M_1 & \omega^2 M_2 & \omega M_2 \\ M_1 & M_1 & M_0 & \omega M_2 & \omega^2 M_2 \\ M_2 & \omega^2 M_2 & \omega M_2 & M_N & M_4 \\ M_2 & \omega M_2 & \omega^2 M_2 & M_4 & M_N' \end{pmatrix}, \quad M_N \equiv y_3^* v_\chi^*, \quad M_N' \equiv y_3^* v_\chi. \tag{2.10}$$

We now derive the effective light-neutrino Majorana mass terms

$$\mathcal{L}_\nu = \frac{1}{2}\left(\nu_{eL}^T\ \nu_{\mu L}^T\ \nu_{\tau L}^T\right) C^{-1} \mathcal{M}_\nu \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} + \text{H.c.}, \tag{2.11}$$

where

$$\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D \tag{2.12}$$

according to the seesaw formula [10]. Because of the special form of M_D in equation (2.7), only the 3×3 upper-left submatrix of M_R^{-1} matters. One finds (for details see appendix A)

$$\mathcal{M}_\nu = \begin{pmatrix} x + y + t & z + \omega^2 y + \omega t & z + \omega y + \omega^2 t \\ z + \omega^2 y + \omega t & x + \omega y + \omega^2 t & z + y + t \\ z + \omega y + \omega^2 t & z + y + t & x + \omega^2 y + \omega t \end{pmatrix}. \tag{2.13}$$

Equations (A.7), (A.8) with $M_2 = M_3$ tell us that

$$(y, t) \propto \left(M_N', M_N\right). \tag{2.14}$$

Therefore, $y/t = v_\chi / v_\chi^*$. We now make the crucial assumption that the VEV v_χ is real. This is *not* an unjustified assumption since it simply corresponds to the conservation of the symmetry $I_{\mu\tau}$ by the VEV of χ. It follows from this assumption that $t = y$, hence

$$\mathcal{M}_\nu = \begin{pmatrix} x + 2y & z - y & z - y \\ z - y & x - y & z + 2y \\ z - y & z + 2y & x - y \end{pmatrix}. \tag{2.15}$$

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This is precisely the $\mathcal{M}_\nu$ corresponding to tri-bimaximal mixing. Its diagonalization reads

$$U_{\mathrm{HPS}}^T \mathcal{M}_\nu U_{\mathrm{HPS}} = \mathrm{diag}\left(\mu_1, \mu_2, \mu_3\right), \tag{2.16a}$$

$$\mu_1 = x + 3y - z, \tag{2.16b}$$

$$\mu_2 = x + 2z, \tag{2.16c}$$

$$\mu_3 = x - 3y - z. \tag{2.16d}$$

The light-neutrino masses are given by $m_j = |\mu_j|$ $(j = 1, 2, 3)$. The matrix $\mathcal{M}_\nu$ has five parameters, corresponding to the three neutrino masses and the two Majorana phases, which are completely free.

2.3 Scalar potential

We have demonstrated that our model leads, under the sole assumption that the VEV v_χ is real, to HPS mixing. In order to check that a real v_χ is viable, we proceed to analyze the scalar potential V of the ϕ_m $(m = 0, e, \mu, \tau)$ and χ. The potential must respect both the three symmetries $\mathbb{Z}_2^{(\alpha)}$ and the permutation symmetry S_3, except for the dimension-two and dimension-one terms, which are allowed to break softly both the $\mathbb{Z}_2^{(\alpha)}$ and $C_{e\mu\tau}$, but not $I_{\mu\tau}$. Therefore,

$$V = \lambda_1 \left[\left(\phi_e^\dagger\phi_e\right)^2 + \left(\phi_\mu^\dagger\phi_\mu\right)^2 + \left(\phi_\tau^\dagger\phi_\tau\right)^2\right] + \lambda_2\left(\phi_0^\dagger\phi_0\right)^2 \tag{2.17a}$$

$$+\lambda_3\left(\phi_e^\dagger\phi_e\,\phi_\mu^\dagger\phi_\mu + \phi_\mu^\dagger\phi_\mu\,\phi_\tau^\dagger\phi_\tau + \phi_\tau^\dagger\phi_\tau\,\phi_e^\dagger\phi_e\right) \tag{2.17b}$$

$$+\lambda_4\,\phi_0^\dagger\phi_0\left(\phi_e^\dagger\phi_e + \phi_\mu^\dagger\phi_\mu + \phi_\tau^\dagger\phi_\tau\right) \tag{2.17c}$$

$$+\lambda_5\left(\phi_e^\dagger\phi_\mu\,\phi_\mu^\dagger\phi_e + \phi_\mu^\dagger\phi_\tau\,\phi_\tau^\dagger\phi_\mu + \phi_\tau^\dagger\phi_e\,\phi_e^\dagger\phi_\tau\right) \tag{2.17d}$$

$$+\lambda_6\,\phi_0^\dagger\left(\phi_e\phi_e^\dagger + \phi_\mu\phi_\mu^\dagger + \phi_\tau\phi_\tau^\dagger\right)\phi_0 \tag{2.17e}$$

$$+\lambda_7\left[\left(\phi_e^\dagger\phi_\mu\right)^2 + \left(\phi_\mu^\dagger\phi_\tau\right)^2 + \left(\phi_\tau^\dagger\phi_e\right)^2 + \mathrm{H.c.}\right] \tag{2.17f}$$

$$+\left\{\lambda_8\left[\left(\phi_0^\dagger\phi_e\right)^2 + \left(\phi_0^\dagger\phi_\mu\right)^2 + \left(\phi_0^\dagger\phi_\tau\right)^2\right] + \mathrm{H.c.}\right\} \tag{2.17g}$$

$$+\left[\lambda_9\left(\phi_e^\dagger\phi_e + \phi_\mu^\dagger\phi_\mu + \phi_\tau^\dagger\phi_\tau\right) + \lambda_{10}\phi_0^\dagger\phi_0\right]|\chi|^2 \tag{2.17h}$$

$$+\lambda_{11}|\chi|^4 + \vartheta_1\left(\chi^3 + \chi^{*3}\right) + \mu_1|\chi|^2 + \mu_2\left(\chi^2 + \chi^{*2}\right) + \eta\left(\chi + \chi^*\right) \tag{2.17i}$$

$$+\lambda_{12}\left[\chi^2\left(\phi_e^\dagger\phi_e + \omega^2\phi_\mu^\dagger\phi_\mu + \omega\phi_\tau^\dagger\phi_\tau\right) + \chi^{*2}\left(\phi_e^\dagger\phi_e + \omega\phi_\mu^\dagger\phi_\mu + \omega^2\phi_\tau^\dagger\phi_\tau\right)\right] \tag{2.17j}$$

$$+\vartheta_2\left[\chi\left(\phi_e^\dagger\phi_e + \omega\phi_\mu^\dagger\phi_\mu + \omega^2\phi_\tau^\dagger\phi_\tau\right) + \chi^*\left(\phi_e^\dagger\phi_e + \omega^2\phi_\mu^\dagger\phi_\mu + \omega\phi_\tau^\dagger\phi_\tau\right)\right] \tag{2.17k}$$

$$+\begin{pmatrix}\phi_0^\dagger & \phi_e^\dagger & \phi_\mu^\dagger & \phi_\tau^\dagger\end{pmatrix}\begin{pmatrix}\mu_3 & \mu_9 & \mu_8 & \mu_8\\ \mu_9^* & \mu_4 & \mu_7 & \mu_7\\ \mu_8^* & \mu_7^* & \mu_5 & \mu_6\\ \mu_8^* & \mu_7^* & \mu_6 & \mu_5\end{pmatrix}\begin{pmatrix}\phi_0\\ \phi_e\\ \phi_\mu\\ \phi_\tau\end{pmatrix}. \tag{2.17l}$$

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The only parameters in V which may be complex are λ_8 and $\mu_{7,8,9}$. Notice the terms μ_2 and η in line (2.17i), which break $C_{e\mu\tau}$ softly, and various terms in line (2.17l) which break the $\mathbb{Z}_2^{(\alpha)}$ (and $C_{e\mu\tau}$) softly. All these terms, though, preserve $I_{\mu\tau}$. The soft breaking of the $\mathbb{Z}_2^{(\alpha)}$ in line (2.17l) is needed in order to prevent the appearance of Goldstone bosons if $\lambda_7 = \lambda_8 = 0$ (see later).

We want both v_χ and the mass of χ to be at the high (seesaw) scale, while both the v_m and the masses of the ϕ_m components should be at the low (electroweak) scale. Therefore we must fine-tune λ_{12} and ϑ_2 in lines (2.17j) and (2.17k), respectively, to be extremely small, lest they pull the masses of the ϕ_α components up to the seesaw scale.[4] Once λ_{12} and ϑ_2 have been tuned to be very small, the phase of v_χ becomes determined only by the terms in line (2.17i). It is clear that, if μ_2 is chosen negative and the product $\vartheta_1\eta$ is chosen positive, then the minimum of V will be obtained for a real v_χ, with sign opposite to the one of ϑ_1 and η [9]. We have thus shown that there is a range of parameters of the scalar potential for which the symmetry $I_{\mu\tau}$ is preserved by the seesaw-scale vacuum, i.e. for which v_χ is real.

At low scale $I_{\mu\tau}$ is spontaneously broken by $|v_\mu| \neq |v_\tau|$. Writing

$$(|v_\mu|\,, |v_\tau|) \propto (\sin\theta, \cos\theta)\,,$$

and assuming all VEVs and coupling constants to be real, we verify that the vacuum potential is, as a function of θ, of the form

$$a + b\,\sin^2 2\theta + c\,\sin 2\theta + d\,\sqrt{1+\sin 2\theta},$$

where $c \propto \mu_6$ and d stems from the $\mu_{7,8}$ terms. Is it clear that a vacuum potential of this form in general leads to a non-trivial value of θ, which may moreover be very small if both c and d are chosen much smaller than $b > 0$.

3 Variations on the symmetries and renormalization-group invariance

The group structure of the model: *all* the symmetries of the model, and their respective breaking mechanisms, have been listed in section 2.1, and in principle it is not necessary to detail the group that they generate. Still, elucidating the group structure of the model may be useful for understanding the terms allowed in the Lagrangian. Following for instance the reasoning in [14], the symmetry group G of our model may be described as the semidirect product

$$G = (N \times H) \rtimes S_3, \tag{3.1}$$

where

$N = \mathbb{Z}_2^{(e)} \times \mathbb{Z}_2^{(\mu)} \times \mathbb{Z}_2^{(\tau)}$ is generated by the $\mathbb{Z}_2$ symmetries of equation (2.4),

$H = \mathrm{U}(1)_{L_e} \times \mathrm{U}(1)_{L_\mu} \times \mathrm{U}(1)_{L_\tau}$ is generated by the family lepton-number symmetries of equation (2.1) and

[4] This fine-tuning is a weak point of our model, but most (non-supersymmetric) models with a very high scale suffer from the same drawback.

the permutation group S_3 is generated by the cyclic permutation $C_{e\mu\tau}$ of equation (2.2) and the transposition $I_{\mu\tau}$ of equation (2.3).

The semi-direct product is non-trivial since neither the $\mathbb{Z}_2^{(\alpha)}$ nor the $\mathrm{U}(1)_{L_\alpha}$ commute with $C_{e\mu\tau}$ and $I_{\mu\tau}$. The elements of G can be written as triples (n, h, s), where $n \in N$, $h \in H$ and $s \in S_3$. The multiplication law of G is the usual one for semidirect products:

$$(n_1, h_1, s_1)\,(n_2, h_2, s_2) = \left(n_1 s_1 n_2 s_1^{-1}, h_1 s_1 h_2 s_1^{-1}, s_1 s_2\right). \tag{3.2}$$

In terms of 3×3 matrices, n is represented by a diagonal sign matrix, h is represented by a diagonal phase matrix and s is a matrix in the defining triplet representation of S_3. According to section 2.1, the representations of G that we employ in our model are

$$\begin{aligned} 1 &\text{ for } \phi_0, \\ ns &\text{ for } \left(\phi_e, \phi_\mu, \phi_\tau\right), \\ nhs &\text{ for } \left(e_R, \mu_R, \tau_R\right), \\ hs &\text{ for } \left(D_{eL}, D_{\mu L}, D_{\tau L}\right) \text{ and } \left(\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}\right), \\ D_2(s) &\text{ for } \left(\nu_{1R}, \nu_{2R}\right) \text{ and } \left(\chi, \chi^*\right), \end{aligned} \tag{3.3}$$

where the two-dimensional irreducible represention (irrep) of S_3 is denoted $D_2(s)$. It is easy to convince oneself that all the multiplets in the list (3.3) constitute irreps of G.

The group G contains all the family symmetries of the dimension-four terms of the Lagrangian. As discussed in detail in section 2, there is a sequence of soft-breaking steps which can be described as

$$G \overset{\dim 3}{\longrightarrow} N \rtimes S_3 \overset{\dim 2}{\longrightarrow} \mathbb{Z}_2^{(\mu\tau)}, \tag{3.4}$$

where $\mathbb{Z}_2^{(\mu\tau)}$ is the $\mathbb{Z}_2$ group generated by $I_{\mu\tau}$.

The variations on the symmetries in the following paragraphs will only concern the normal subgroup H of G.

The symmetry group $\mathbf{\Delta(6\infty^2)}$: if we remove from the three $\mathrm{U}(1)_{L_\alpha}$ the global $\mathrm{U}(1)_L$ associated with the total lepton number $L = L_e + L_\mu + L_\tau$, then the normal subgroup H of G reduces to the set of matrices

$$U\left(\beta, \gamma\right) = \mathrm{diag}\left(e^{i\beta}, e^{i\gamma}, e^{-i\beta - i\gamma}\right), \quad \beta, \gamma \in [0, 2\pi[\,. \tag{3.5}$$

In this case, $H \rtimes S_3$ is the group $\Delta(6\infty^2)$, or rather a faithful irrep thereof — see [15] for a study of this group. Therefore, $G = N \rtimes \Delta(6\infty^2)$.

Switching to $\mathbf{\Delta(54)}$: $\Delta(54)$ is the group $\Delta(6r^2)$ with $r = 3$ — for details see [15–17].[5] In this variant of our model we do not use the symmetries $\mathrm{U}(1)_{L_\alpha}$. Instead, we define the matrix [8]

$$T \equiv \mathrm{diag}\left(1, \omega, \omega^2\right), \tag{3.6}$$

and use a symmetry under which the multiplets transform according to table 1. The trans-

[5] The latter paper uses $\Delta(54)$ for the construction of a lepton flavour model which is, however, totally different from ours.

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	$D_{\alpha L}$	α_R	$\nu_{\alpha R}$	ϕ_α
T	T	T^*	T	T^2

Table 1. Transformation of the multiplets under the symmetry T. The multiplets not shown in the table transform trivially.

formation T, together with the 3×3 permutation matrices, generates a three-dimensional irrep of $\Delta(54)$. Notice that this group is *a priori* smaller — hence less powerful — than $\Delta(6\infty^2)$, but we enhance its power by allowing it to act non-trivially on the ϕ_α. It is easy to check that the Yukawa Lagrangian of equation (2.5) is invariant under T, but we still need the symmetries $\mathbb{Z}_2^{(\alpha)}$ to remove from $\mathcal{L}_{\text{Yukawa}}$ possible non-flavour-diagonal terms [8]. So the symmetry group of our model is now $G = N \rtimes \Delta(54)$, which is *finite* and has $8 \times 54 = 432$ elements. We may still describe G through equation (3.1), with H replaced by

$$H = \left\{ \text{diag}\left(\omega^p, \omega^q, \omega^{-p-q}\right) \mid p, q = 0, 1, 2 \right\}. \tag{3.7}$$

Concerning the irreps, instead of list (3.3) we now have

$$\begin{aligned} \mathbf{1} \ &\text{for} \ \phi_0, \\ nh^2s \ &\text{for} \ (\phi_e, \phi_\mu, \phi_\tau), \\ nh^*s \ &\text{for} \ (e_R, \mu_R, \tau_R), \\ hs \ &\text{for} \ (D_{eL}, D_{\mu L}, D_{\tau L}) \ \text{and} \ (\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}), \\ D_2(s) \ &\text{for} \ (\nu_{1R}, \nu_{2R}) \ \text{and} \ (\chi, \chi^*). \end{aligned} \tag{3.8}$$

The breaking of T is assumed to be soft, through dimension-three and dimension-two terms. An important difference relative to section 2 is that T removes some of the dimension-four terms from the scalar potential, because it acts non-trivially on the ϕ_α; one obtains a restricted version of equation (2.17), *viz.*

$$\lambda_7 = \lambda_8 = 0. \tag{3.9}$$

Notice that, although we did not use the U_{L_α} in building this variant of the model, eventually the U_{L_α} turn out to be (so-called accidental) symmetries of all the dimension-four terms in the Lagrangian.

Switching to $\boldsymbol{\Delta(6r^2)}$ with $\boldsymbol{r \geq 4}$: if the T of the previous paragraph is replaced by

$$T = \text{diag}\,(1, \sigma, \sigma^*), \quad \sigma \equiv \exp\,(2i\pi/r), \quad r \geq 4, \tag{3.10}$$

then

$$H = \left\{ \text{diag}\left(\sigma^p, \sigma^q, \sigma^{-p-q}\right) \mid p, q = 0, \ldots, r-1 \right\} \tag{3.11}$$

and $H \rtimes S_3$ is isomorphic to $\Delta(6r^2)$ with $r \geq 4$. All the previous remarks, including table 1, still hold in this case, but there is a noteworthy exception: now we do not need to impose the symmetries $\mathbb{Z}_2^{(\alpha)}$, which become just *accidental* symmetries of all the terms in the Lagrangian with dimension larger than two. Eventually, the family symmetry group of the model is again of the form of equation (3.1), with $G \cong N \rtimes \Delta(6r^2)$ having $48r^2$ elements.

Renormalization-group evolution of $\mathcal{M}_\nu$: we proceed to the study of the RGE of $\mathcal{M}_\nu$ from the seesaw scale down to the electroweak scale. We first note that the two real degrees of freedom of the scalar gauge singlet χ are assumed to be heavy. Therefore, the renormalization-group (RG) equations relevant for the determination of $\mathcal{M}_\nu$ at the low scale are simply those of a multi-Higgs-doublet SM. Those equations were derived in [18]. It was shown in [9] that the form of the Yukawa couplings of the charged-lepton fields — see line (2.5a) — remains unchanged; only the value of y_1 evolves with the energy scale. In the same paper [9], the importance of the quartic scalar couplings for the RGE of $\mathcal{M}_\nu$ was investigated; the following sufficient conditions for RG invariance of $\mathcal{M}_\nu$ were found:

i) The Higgs doublet ϕ_0, whose VEV v_0 is responsible for generating $\mathcal{M}_\nu$ at the seesaw scale, has no Yukawa couplings to the α_R. In our model, the Yukawa couplings of the charged leptons are given by line (2.5a) at any energy scale.

ii) There is a symmetry, holding at the seesaw scale, which forbids dimension-five neutrino mass operators involving two different Higgs doublets. In our model, that symmetry is constituted by the three $\mathbb{Z}_2^{(\alpha)}$.

iii) At the seesaw scale there is a symmetry forbidding quartic couplings of the type $\left(\phi_k^\dagger \phi_{k'}\right)^2$ $(k \neq k')$ in the scalar potential. In our model, this is satisfied if some symmetry like T leads to the condition (3.9).

Thus, applying the results of our previous paper [9] to the present model, we find that, if equation (3.9) holds, then *tri-bimaximal mixing holds at all energy scales in between the seesaw and electroweak scales*. According to the preceding discussion, this is possible by using any of the symmetry groups $\Delta(6r^2)$ $(r \geq 3)$. On the other hand, using $\Delta(6\infty^2)$ allows both λ_7 and λ_8 to be non-vanishing, and then corrections to tri-bimaximal mixing from the RGE of $\mathcal{M}_\nu$ are expected. Still, it is well known that such corrections can only be sizable for a quasi-degenerate neutrino mass spectrum [19], an observation corroborated by explicit studies of multi-Higgs doublet models [18] and general considerations [20].

S_3 versus S_4: in a series of papers [21] it has been argued that the only finite group capable of yielding tri-bimaximal mixing is S_4, or else a larger group containing S_4. We want to make some comments on that claim. Since $S_4 \equiv \Delta(24)$ [16], we can expect that a construction of our model in analogy to the usage of $\Delta(6r^2)$ with $r \geq 3$ is possible. This is indeed the case. We can place the $D_{\alpha L}$, the α_R and the $\nu_{\alpha R}$ in triplets of S_4. Putting the ϕ_α in the *reducible* triplet representation of the subgroup S_3 and adding to this scheme the symmetries $\mathbb{Z}_2^{(\alpha)}$ in order to avoid non-flavour-diagonal couplings in $\mathcal{L}_{\text{Yukawa}}$, we can proceed with the construction of the model just as in section 2. Actually, it is easy to see that this way of constructing the model amounts simply to the replacement of the $\mathrm{U}(1)_{L_\alpha}$ by discrete lepton numbers: fermions with flavour α are multiplied by -1 instead of being multiplied by an arbitrary phase factor. In the language of equation (3.1), in this case the family symmetry group is $G = (N \times N) \rtimes S_3$ — for a complete discussion of its irreps see [14]. However, it appears to us that S_4 is not an adequate symmetry group for our model for two reasons. First, the full symmetry group, which is only effective in terms of dimension four in the Lagrangian, is much larger than S_4 because its subgroup S_3 does not commute with the $\mathbb{Z}_2^{(\alpha)}$; therefore, S_4 misses an essential part of the symmetry structure of our model. Second, in the terms of dimension three, i.e. in $\mathcal{L}_{\mathrm{Majorana}}$, which are crucial for our model, the symmetry group is only S_3, something that we had already advocated in [5]. In summary, in our model there is no compelling connection between S_4 and tri-bimaximal mixing.

4 Conclusions

In this paper we have proposed a model for tri-bimaximal mixing based on an extension of the SM with seesaw mechanism and family symmetries. The scalar sector consists of four Higgs doublets and one complex gauge singlet, while the fermion sector has, besides the SM multiplets, five right-handed neutrino singlets. The mixing matrix obtained at the seesaw scale is exactly tri-bimaximal. The most straightforward version of the model uses as family symmetries the permutation group S_3 together with three $\mathbb{Z}_2$ symmetries and family lepton numbers; the latter are softly broken at the seesaw scale. A slightly more complicated way to obtain the model makes use of a group $\Delta(6r^2)$ with $r \geq 3$. The most intricate part of the model is the stepwise soft symmetry breaking, which we have tried to explain carefully in section 2. Whether one uses S_3 together with family lepton numbers or a group $\Delta(6r^2)$ does not make any difference, except for two terms of dimension four in the scalar potential. With $\Delta(6r^2)$ those two terms are forbidden and, as a consequence, in the one-loop renormalization-group evolution of the neutrino mass matrix from the seesaw scale down to the electroweak scale, that matrix retains its form and tri-bimaximal mixing remains exact at the electroweak scale. With S_3 together with family lepton numbers there are the usual RGE corrections, which are quite small, however, whenever the neutrino mass spectrum is sufficiently non-degenerate.

The main purpose of the model presented here is to show that in enforcing tri-bimaximal mixing one does not necessarily require VEV alignment, supersymmetry, non-renormalizable terms, or extra dimensions. As a further bonus, one can also obtain RG stability of HPS mixing.

Finally, we want to stress that in our model there is decoupling of the mixing problem from the mass problem; the latter remains unsolved, since all lepton masses are completely free.

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A Inverting M_R

The 5×5 symmetric matrix

$$M = \begin{pmatrix} M_0 & M_1 & M_1 & M_2 & M_3 \\ M_1 & M_0 & M_1 & \omega^2 M_2 & \omega M_3 \\ M_1 & M_1 & M_0 & \omega M_2 & \omega^2 M_3 \\ M_2 & \omega^2 M_2 & \omega M_2 & M_N & M_4 \\ M_3 & \omega M_3 & \omega^2 M_3 & M_4 & M'_N \end{pmatrix}, \quad \omega \equiv \exp(2i\pi/3) \tag{A.1}$$

has non-zero determinant:

$$\det M = (M_0 + 2M_1) \left\{ (M_0 - M_1)^2 M_N M'_N - [(M_0 - M_1) M_4 - 3M_2 M_3]^2 \right\}. \tag{A.2}$$

Let us write

$$M^{-1} = \begin{pmatrix} P & R \\ R^T & Q \end{pmatrix}, \tag{A.3}$$

where R is a 3×2 matrix and Q is a 2×2 symmetric matrix. Then,

$$P = \begin{pmatrix} x + y + t & z + \omega^2 y + \omega t & z + \omega y + \omega^2 t \\ z + \omega^2 y + \omega t & x + \omega y + \omega^2 t & z + y + t \\ z + \omega y + \omega^2 t & z + y + t & x + \omega^2 y + \omega t \end{pmatrix}, \tag{A.4}$$

with

$$x = \frac{\left(M_0^2 - M_1^2\right)\left(M_N M'_N - M_4^2\right) + (4M_0 + 2M_1) M_2 M_3 M_4 - 3M_2^2 M_3^2}{\det M}, \tag{A.5}$$

$$z = \frac{\left(M_1^2 - M_0 M_1\right)\left(M_N M'_N - M_4^2\right) + (M_0 - 4M_1) M_2 M_3 M_4 - 3M_2^2 M_3^2}{\det M}, \tag{A.6}$$

$$y = \frac{(M_0 + 2M_1) M_2^2 M'_N}{\det M}, \tag{A.7}$$

$$t = \frac{(M_0 + 2M_1) M_3^2 M_N}{\det M}. \tag{A.8}$$

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